MAY

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Find the number of seven-digit numbers such that every number is divisible by 125, no number contains a zero, and no digit appears more than once in the number.

1

For all integers a, when ka is subtracted from $49a^2 + 16$, the result is a perfect square. Find the value or values of k.

2

Let T_n be a triangle with vertices

$$(2n, 1), (2n + 2, 1), \text{ and } (-4n, -3).$$

Find the sum of the areas of the set of triangles

$$\{T_1, T_2, T_3, \ldots, T_{100}\}.$$

3

A, B, C, and D represent distinct digits, and ABC represents a three-digit number, *not* the product $A \cdot B \cdot C$. If

$$ABC^C = DABBBAB,$$

find the number $ABCD$.

4

For how many positive integers

$$k \le 2010$$

does division of k^2 by 4 leave a remainder of 1?

5

To convert Celsius temperatures to Fahrenheit temperatures quickly without a calculator, some people use this rule of thumb: Double the Celsius reading and add 30°. The correct formula is, of course, $F = (9/5)C + 32$. What is the largest error made using the rule of thumb for Celsius temperatures between −10° and 35°, inclusive?

6

Define

$$S_n = 1^7 + 2^7 + 3^7 + \cdots + n^7$$

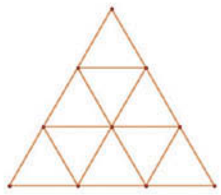
for all natural numbers n. Write the expression

$$\log_{\sqrt{7}}\left(S_{343} - S_{342}\right)$$

in simplest form.

7

Eighteen toothpicks are arranged to create exactly 13 triangles of various sizes. (Ignore other polygons.) Remove 4 toothpicks so that exactly 5 triangles remain.



8

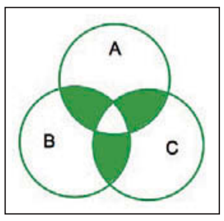
Three pennies showing 2 heads and 1 tail lie on a table. Two of the coins are selected at random and turned over. What is the probability that now all three coins show the same side?

9

Find the integer base for which the base-10 number 7642 is written as 1234.

10

Sets A, B, and C are indicated in the Venn diagram. Use standard notation for union, intersection, and complement to write a description of the shaded subset.

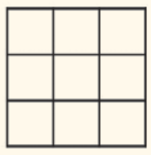


11

How many distinct isosceles triangles exist such that the sides have integral length and the perimeter is 113?

12

Three darts are thrown at the square dartboard shown, and each lands in a different small square. What is the probability that the 3 squares in which the 3 darts land form a horizontal, vertical, or diagonal row?



13

Given

$$f(x) = x + 1$$

and

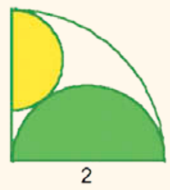
$$F(x, y) = x^2 + y,$$

find the largest integer x such that

$$F(f(x), f(x)) \le 2010.$$

14

The two semicircles in the figure are tangent to each other. The radius of the quadrant and the diameter of the large semicircle are each 2. Find the diameter of the smaller semicircle.



15

Country A has c% of the world's population and owns d% of the world's wealth. Country B has e% of the world's population and owns f% of its wealth. Suppose the citizens of each country share the wealth of their country equally. Find the ratio of the wealth of a citizen of A to the wealth of a citizen of B.

16

What is the smallest integer greater than 1 that is simultaneously a perfect square, a perfect cube, and a fifth power?

17

In a singing contest there are 6 finalists: Ari, Bella, Calli, Della, Ella, and Fitz. Ari will perform before Bella. What is the probability that Ari will perform immediately before Bella?

18

If

$$i^2 = -1,$$

for how many integers n is

$$(n + i)^4$$

an integer?

19

An old video game featured a hungry creature in the form of a sector of a circle with radius 1 cm. The missing piece, the mouth, has a central angle of 60°. What is the perimeter (in cm) of the creature?

20

Solve for x over the set of real numbers:

$$\left(2^x - 4\right)^3 + \left(4^x - 2\right)^3 = \left(2^x + 4^x - 6\right)^3$$

21

Given the functions

$$f(x) = (1/2)x^2 \text{ and } g(x) = kx,$$

find the value of k so that the amount of change in f as x increases from −4 to −1 equals the amount of change in g as x increases from −4 to −1.

22

Ana and Bob play a game in which Ana begins by rolling a fair die, after which Bob tosses a fair coin. They take turns until one of them wins. Ana wins when she rolls a 6; Bob wins when the coin lands heads. What is the probability that Ana wins the game?

23

Find the average of all multiples of 7 between 7 and 777, inclusive.

24

What is the greatest possible remainder when a two-digit integer is divided by the sum of its digits?

25

What is the length of the diagonal of an isosceles trapezoid with side lengths 7, 8, 8, and 15?

26

If

$$x + x^{-1} = 3,$$
$$x^2 + x^{-2} = a,$$

and

$$x^3 + x^{-3} = b,$$

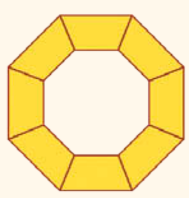
find the numerical value of $a + b$.

27

The sides of a right triangle with integral lengths are in arithmetic progression. What are the side lengths of the smallest such triangle whose perimeter is greater than 2010?

28

Eight identical tables shaped as isosceles trapezoids fit together without gaps to make a "round" table. Find the interior angle measures of the trapezoidal tabletops.



29

Consider the set of positive integer divisors of 2010. What is the probability that a divisor selected at random is even?

30

The integer 2010 cannot be written as the sum of 2 squares but can be written as the sum of 3 squares in more than 5 ways. Find as many of the different ways as you can.

31